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Free convection flow of elastico-viscous liquid from horizontal plate

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The free convection flow of an elastico-viscous liquid from a horizontal plate has been studied in this paper The type of wall temperature distribution and the boundary layer thickness which allow similarity solution are investigated. The effect of elasticity of the liquid and the Prandtl number on the velocity and temperature distributions and the rate of heat transfer from the plate has been studied.

INTRODUCTION

The phenomenon of natural convection arises in a fluid when temperature changes cause density variations leading to buoyancy forces acting on the fluid elements. This process of heat transfer has many important technological applications. Therefore, in modern times there has been a noticeable increase in interest in free convection problems. Studies on laminar free convection flow and heat transfer of Newtonian fluids have been reported in literature. But little work in this direction seems to have been done in the case of non-Newtonian fluid. Recently, Mishra (1966) has studied the free convection flow of an elastico-viscous liquid past a hot vertical plate In this paper our aim is to study the free convection flow of an elastico-viscous liquid from a horizontal plate. Similar problems in the Newtonian case have been studied by Sparrow & Minkowycz (1962), Gill & Casal (1962), Hauptmann (1965), Gill *et al* (1965). This problem in the presence of a magnetic field has been studied by Gupta (1966). Free convection flow of a second order fluid from a horizontal plate has been studied by Mishra (1968).

BASIC EQUATIONS

The equations governing the elastico-viscous fluid model considered here consist of the constitutive equations

$$p_{ik} = -pg_{ik}+p'_{ik}, \qquad \dots \quad (1)$$

$$p'^{ik} = 2\eta_0 e^{ik} - 2k_0 \overset{\smile}{e}^{ik}. \qquad \dots \quad (2)$$

The equations of motion and continuity are

$$\rho\left[\frac{\partial v^i}{\partial t}+v^j v^i{}_{,j}\right]=F_i-p_{,i}+p'_{,ij}{}^{ij} \qquad \ldots \quad (3)$$

and

$$v_{i,i}=0. \qquad \ldots \quad (4)$$

In these equations the term $\tilde{e}^{ik}$ appearing in (2) is given by

$$\tilde{e}^{ik}=\frac{\partial e^{ik}}{\partial t}+v^m e_{,m}{}^{ik}-v_{,m}{}^{k}e^{im}-v_{,m}{}^{i}e^{mk}$$

is the convected derivative of the rate-of-strain tensor e^{ik} and is defined as

$$2e_{ik}=v_{i,k}+v_{k,i}\ ,$$

v^i being the velocity vector, p is the mean pressure, ρ is the density of the medium, F_i is the body force per unit mass in the i-th direction and p_{ik} is the stress tensor. The limiting viscosity at small rates of shear is

$$\eta_0=\int_0^\infty N(\lambda)d\lambda$$

$N(\lambda)$ being the relaxation spectrum as introduced by Walters (1960), and g_{ik} is the metric tensor of a fixed coordinate system x^i. This idealized model (2) is a valid approximation of Walter's liquid B' taking very short memory into account such that terms involving

$$\int_0^\infty \lambda^n N(\lambda)d\lambda \qquad (n\geq 2)$$

have been neglected and

$$k_0=\int_0^\infty \lambda N(\lambda)d\lambda.$$

A detailed description of the model has been given by Walters & Beard (1964) The energy equation, neglecting the dissipation term, which is justified for slow motion as in the case of free convection flow, is

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}, \qquad \ldots \quad (5)$$

where T is the temperature and α is the thermal diffusivity.

Boundary Layer Equations

In free convection problems, the thickness of the layer, in which the temperature and velocity differ appreciably from the values at infinity, is found to be small compared with the length of the plate; hence the approximation of the boundary layer theory will be valid Recently, Beard & Walters (1964) have obtained the boundary layer equations for this class of fluid. Within the boundary layer $u, \partial u/\partial x, \partial^2 u/\partial x^2, \partial p/\partial x$ are assumed to be of the order unity and y to be of the order of the boundary layer thickness. From the equation of continuity (4) we get the y-component of the velocity to be of the order of the boundary layer thickness. In order that the viscous, elastico-viscous and inertia terms in the modified equations of motion be of the same order of magnitude, it is necessary that

$$\nu = 0(\delta^2) \quad \text{and} \quad k_0{}^* = 0(\delta^2)$$

where

$$\nu = \eta_0/\rho \quad \text{and} \quad k_0{}^* = k_0/\rho.$$

Under these conditions, equations of motion give

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} - k_0{}^*\left[\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}+v\frac{\partial^3 u}{\partial y^3} + \frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right)\right], \quad \ldots (7)$$

$$0 = F_y - 1/\rho.\ \partial p/\partial y, \quad \ldots (8)$$

with the equation of continuity $\partial u/\partial x+\partial v/\partial y = 0$ ... (8a)

In the present problem $F_x = 0$ and $F_y = -g$, the acceleration due to gravity, since we study free convection flow from a horizontal plate We are concerned with the velocity and temperature distributions in the boundary layer over the plate

Solution of Equations

Since the ambient fluid is at rest, from equation (7) we get

$$\frac{\partial p_\infty}{\partial x} = 0, \quad \ldots (9)$$

where the subscript ∞ refers to ambient state. The equation of state is

$$\rho = \rho_\infty[1-\beta(T-T_\infty)], \quad \ldots (10)$$

where β, the coefficient of volume expansion is assumed constant Hence equation (8) gives

$$-\frac{\partial p}{\partial y} = \rho_\infty g[1-\beta(T-T_\infty)], \quad \ldots (11)$$

which when differentiated with respect to x gives

$$\frac{\partial^2 p}{\partial x \partial y} = \rho_\infty g\beta \frac{\partial}{\partial x}(T-T_\infty). \quad \ldots (12)$$

Integration of equation (12) with respect to y from y to δ and subsequent use of equation (8) yields.

$$\frac{\partial p}{\partial x} = -\rho_\infty g\beta \int_y^\delta \frac{\partial}{\partial x}(T-T_\infty)dy, \quad \ldots (13)$$

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where condition at infinity is replaced by the condition at the outer edge of the boundary layer Substitution of equation (13) in equation (7) leads to

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\beta g\frac{\partial}{\partial x}\int_y^\delta (T-T_\infty)dy+\nu\frac{\partial^2 u}{\partial y^2}$$

$$-k_0^*\left[\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}+v\frac{\partial^3 u}{\partial y^3}+\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right)\right], \quad \dots \quad (14)$$

where $\nu=\eta_0/\rho_\infty$. This, along with equation (5) and (6) constitute the basic equations of the problem which are to be solved subject to

$$u=v=0 \quad T=T_w \quad \text{at} \quad y=0 \quad \dots \quad (15)$$

$$u=\frac{\partial u}{\partial y}=0, \quad T=T_\infty \quad \text{at} \quad y=\delta \quad \dots \quad (16)$$

We solve the boundary layer equation and energy equation by an integral method similar to that of Karman & Pohlhausen. Integration of equation (14) with respect to y from 0 to δ and use of equation (6) lead to

$$\frac{\partial}{\partial x}\int_0^\delta u^2dy=-\nu\left(\frac{\partial u}{\partial y}\right)_w+\beta g\frac{\partial}{\partial x}\left[(T_w-T_\infty)\int_0^\delta\frac{(T-T_\infty)}{(T_w-T_\infty)}\right]dy$$

$$k_0^*-\left[\frac{\partial}{\partial x}\int_0^\delta\left(\frac{\partial u}{\partial y}\right)^2dy+\left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}\right)_w+\left(v\frac{\partial^2 u}{\partial y^2}\right)_\delta+2\int_0^\delta\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}dy\right] \quad (17)$$

Assuming

$$\theta=\frac{T-T_\infty}{T_w-T_\infty} \quad \text{and} \quad T_w-T_\infty=Nx^n, \quad (18)$$

equation (5) can be written as

$$u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}+\frac{nu\theta}{x}=\alpha\frac{\partial^2\theta}{\partial y^2} \quad (19)$$

which when integrated with with respect to y from 0 to δ gives

$$\frac{\partial}{\partial x}\int_0^\delta u\theta dy=-\alpha\frac{\partial\theta}{\partial y}-\frac{n}{x}\int_0^\delta u\theta dy \quad (20)$$

We take simple profiles for u and θ satisfying equations (15) and (16) as

$$u=u_1(x)\frac{y}{\delta}\left[1-\frac{y}{\delta}\right]^2 \quad (21)$$

$$\theta=\left[1-\frac{y}{\delta}\right]^2. \quad (22)$$

Substitution of equations (21) and (22) leads to

$$\frac{d}{dx}\left[\frac{\delta u_1^2}{105}\right] = -\frac{\nu u_1}{\delta}+\frac{1}{12}\beta g N\frac{d}{dx}\left[\ x^n\delta^2\ \right]-\frac{k_0^*}{10}\left[\frac{u_1^2}{\delta}\ \frac{d\delta}{dx}-\frac{7u_1}{\delta}\ \frac{du_1}{dx}\right] \quad \ldots \quad (23)$$

and

$$\frac{1}{30}\left[\ \delta\ \frac{du_1}{dx}+u_1\frac{d\delta}{dx}\right] = \frac{2\alpha}{\delta}-\frac{nu_1\delta}{30x} \quad \ldots(24)$$

Let

$$u_1 = C_1x^m \quad \text{and} \quad \delta = C_2x^l. \quad \ldots \quad (25)$$

so that equations (23) and (24) become

$$\frac{C_1^2C_2}{105}(l+2m)x^{2m+l-1} = -\frac{\nu C_1}{C_2}\ x^{m-l}+\frac{1}{12}\ \beta g N C_2^2(2l+n)x^{2l+n-1}$$

$$-\ \frac{k_0^*}{10}\ \ \frac{C_1^2}{C_2}\,(l-7m)x^{2m-l-1} \quad \ldots \quad (26)$$

$$\frac{1}{30}C_1C_2(l+m)x^{l+m-1} = \frac{2\alpha}{C_2}\ x^{-l}\ -\frac{nC_1C_2}{30}\ x^{l+m-1}. \quad \ldots \quad (27)$$

Since these equations are valid for all values of x

$$\left.\begin{aligned} 2m+l-1 = m-l = 2l+n-1 = 2m-l-1 \\ l+m-1 = -l \end{aligned}\right\} \quad \ldots \quad (28)$$

which leads to

$$m = 1, \quad l = 0, \quad n = 2. \quad \ldots \quad (29)$$

It is interesting to note that a similarity solution is possible only if the difference between the wall and ambient temperatures varies as the square of the distance from the leading edge along the plate and the boundary layer thickness is uniform all through. Substituting the values of l, m and n into equations (26) and (27) we get

$$\frac{2}{105}\ C_1^2C_2 = -\ \frac{\nu C_1}{C_2}\ +\frac{1}{6}\beta g N C_2^2+\frac{7k_0^*C_1^2,}{10C_2}\ , \quad (30)$$

$$C_1C_2^2 = 20\alpha. \quad (31)$$

Elimination of C_1 between (30) and (31) gives

$$\lambda^7-(21P_r+8)\lambda^2+R_c = 0 \quad (32)$$

whe

$$\lambda = \left[\frac{7\beta g N}{40\alpha^2}\right]^{1/5} C_2;\ R_c = 294k_0^*\ \left[\frac{7\beta g N}{40\alpha^2}\right]^{2/5}$$

and P_r is the Prandtl number ν/α. We take in this problem $P_r = 1/3$, 2, 12, and 32 so that the equation (32) reduces to

$$\lambda^7 - 15\lambda^2 + R_c = 0 \quad \ldots \quad (32a)$$

$$\lambda^7 - 50\lambda^2 + R_c = 0 \quad \ldots \quad (33b)$$

$$\lambda^7 - 260\lambda^2 + R_c = 0 \quad \ldots \quad (33c)$$

$$\lambda^7 - 680\lambda^2 + R_c = 0 \quad \ldots \quad (33d)$$

These equations can have at best two positive roots and one negative root and four roots are imaginary. Negative and the imaginary roots are not to be considered Table 1 shows the values of λ for different values of R_c and P_r. For the viscous case one root is 0 00 which is discarded since the boundary layer thickness is not zero and the roots in the elastico-viscous case which are in correspondence with this root are therefore discarded. Table 1 shows that the boundary layer thickness

TABLE 1. Values of λ for different values of P_r and R_c.

R_c \ P_r	1/3	2	12	32
0 0	1.7187	2.2034	3 0174	3.6944
0 5	1.7123	2 2021	3.0173	3.6943
1 0	1.7062	2.2009	3.0172	3.6943
1.5	1.7035	2.1996	3.0170	3 6942

decreases with the increase in the value of the elastic number This result agrees with the result by Beard & Walters (1964) but contradicts that by Rajeswari & Rathna (1962) This is because in the Rivlin-Ericksen fluid model the latter authors have taken the memory coefficient to be positive which should be, in fact, negative, as has been proved by Coleman & Markovitz (1964). Davis (1966) has also made the samé remark in his paper. But the boundary layer thickness increases with the increase in the value of Prandtl number. With the help of (31) we have

$$\frac{C_1}{20\alpha}\left[\frac{40\alpha^2}{7\beta gN}\right]^{2/5} = C_1{*} = \frac{1}{\lambda_2}$$

Table 2 shows that the value of $C_1{*}$ goes on increasing with the increase in elastic number but decreases as the Prandtl number increases.

Equations (21) and (22) can be written in the forms

$$\frac{u^*}{x^*} = \frac{y^*}{\lambda^3}\left[1-\frac{y^*}{\lambda}\right]^2 \quad \text{and} \quad \theta = \left[1-\frac{y^*}{\lambda}\right]^2 \quad \ldots \quad (34)$$

Table 2. Values of C_1^* for different values of P_r and R_0

R_0 \ P_r	1/3	2	12	92
0.0	0.3385	0 2059	0 1098	0.0734
0.5	0.3411	0 2062	0 1098	0.0734
1.0	0.3435	0.2064	0 1098	0 0734
1 5	0.3447	0 2066	0 1099	0 0735

where

$$x^* = x\left[\frac{7\beta gN}{40\alpha^2}\right]^{1/5};\ y^* = y\left[\frac{7\beta gN}{40\alpha^2}\right]^{1/5}\ u^* = \left(\frac{u}{20\alpha}\right)\left[\frac{40\alpha^2}{7\beta gN}\right]^{1/5}$$

The rate of heat transfer at the wall is given by

$$q = -k\left(\frac{\partial T}{\partial y}\right)_w = \frac{2k(T_w - T_\infty)}{C_2} = \frac{2kNx^2}{C_2}$$

so that the Nusselt number $N_{ux} = 2x/C_2 = 2x^*/\lambda$. ... (35)

Since λ decreases with the increase in the value of the elastic number, the Nusselt number clearly increases with the increase of the elasticity of the liquid. Also, since λ increases with the increase in the value of P_r, the Nusselt number decreases as Prandtl number increases Table 3 represents the computed values of u^*/x^* and θ. This table shows that the velocity at any point within a thin liquid layer near the plate increases with the increase in the value of the elastic number. But outside this layer up to the edge of the boundary layer the velocity decreases with the increase in the value of the elastic number This nature of the profile may be explained from a consideration of the conservation of the mass flux. Since the velocity increases in a thin layer near the plate due to the elasticity of the liquid, the velocity should decrease outside this layer. It can easily be seen from equation (34) that the minimum value of u^* occurs at $y = 0$ and at $y = \lambda$, that is, at the plate and at the edge of the boundary layer The maximum value of u^* occurs at $\lambda/3$ and its value is $4/27\ \lambda^2$. Figure 1 shows that the velocity at any point in the boundary layer decreases with the increase in the value of the Prandtl number. Table 4 shows that the elasticity of the liquid decreases with the temperature at any point in the boundary layer. Figure 2 shows that the temperature at any point in the boundary layer increases with the increase in the value of the Prandtl number.

TABLE 3. Velocity distribution for different values of R_c. $P_r = 2$

y^* \ R_c	0.00	0.50	1.00
0.00	0.0000	0.0000	0.0000
0.10	0.0084	0.0085	0.0085
0.20	0.0153	0.0154	0.0155
0.30	0.0208	0.0209	0.0210
0.40	0.0249	0.0250	0.0251
0.50	0.0278	0.0279	0.0280
0.60	0.0295	0.0297	0.0298
0.70	0.0303	0.0305	0.0306
0.80	0.0302	0.0303	0.0304
0.90	0.0293	0.0294	0.0295
1.00	0.0278	0.0279	0.0280
1.10	0.0257	0.0258	0.0258
1.20	0.0232	0.0235	0.0233
1.30	0.0204	0.0204	0.0204
1.40	0.0174	0.0173	0.0173
1.50	0.0143	0.0142	0.0142
1.60	0.0112	0.0100	0.0100
1.70	0.0101	0.0091	0.0089
1.80	0.0080	0.0067	0.0062
1.90	0.0052	0.0043	0.0031
2.00	0.0016	0.0009	0.0006

TABLE 4. Temperature distribution for different values of R_c. $P_r = 2$.

y^* \ R_c	0.0	0.5	1.0	1.5
0.0	1.0000	1.0000	1.0000	1.10000
0.1	0.9114	0.9112	0.9112	0.9110
0.2	0.8270	0.8266	0.8265	0.8262
0.3	0.7466	0.7461	0.7460	0.7456
0.4	0.6704	0.6697	0.6695	0.6691
0.5	0.5983	0.5975	0.5973	0.5967
0.6	0.5302	0.5294	0.5291	0.5285
0.7	0.4662	0.4653	0.4649	0.4644
0.8	0.4064	0.4055	0.4052	0.4044
0.9	0.3508	0.3497	0.3493	0.3486
1.0	0.2992	0.2981	0.2977	0.2970
1.1	0.2517	0.2506	0.2501	0.2495
1.2	0.2083	0.2072	0.2067	0.2061
1.3	0.1690	0.1676	0.1674	0.1668
1.4	0.1338	0.1327	0.1322	0.1317
1.5	0.1027	0.1017	0.1014	0.1008

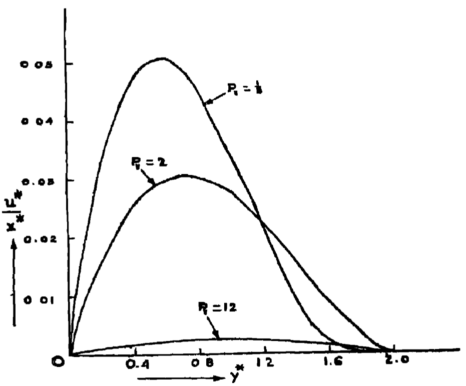

Figure 1. Velocity profiles for different Prandtl numbers.

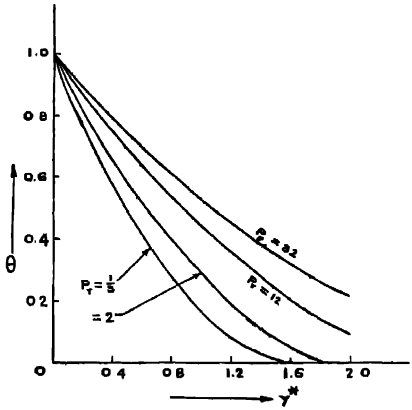

Figure 2 Temperature profiles for different Prandtl numbers.

Gupta (1966) has studied the problem of hydromagnetic free convection flows from a horizontal plate and has obtained the similarity solutions for the velocity and temperature fields His conclusions are, for a uniform magnitic field

$$T_w - T_\infty = Nx^2; \quad u_1(x) = C_1 x\ ; \quad \delta = C_2$$

which we have obtained in our problem in (18), (25) and (29) This gives us an interesting result that the elasticity of the liquid plays some role similar to a magntic field present at the plate. In this connection reference may also be made to another paper by Gupta (1962).

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